Math 3210 Tutorial 2

Example 0: How to solve a linear Problem:

Max : $Z = 4x_1 - 2x_2 + 7x_3$ subject to

$2x_1 - x_2 + 4x_3 \le 18$

$4x_1 + 2x_2 + 5x_3 \le 10$

$x_1, x_2, x_3 \ge 0$

step1: add variables

$2x_1 - x_2 + 4x_3 + s_1 = 18$

$4x_1 + 2x_2 + 5x_3 + 0s_1 + s_2 = 10$

$x_1, x_2, x_3, s_1, s_2 \ge 0$

step2: turn it into matrix from

$$\begin{pmatrix} 2 & -1 & 4 & 1 & 0 \\ 4 & 2 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 18 \\ 10 \end{pmatrix}$$

step 3-ϕ: find all basic solution

i) $x_3 = s_1 = s_2 = 0$ $\begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 18 \\ 10 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cancel{\ } \\ \cancel{\ } \end{pmatrix} \begin{pmatrix} 23/4 \\ -\frac{13}{2} \end{pmatrix}$ → out of feasible region.

ii) $x_2 = s_1 = s_2 = 0$ $\begin{pmatrix} 2 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 18 \\ 10 \end{pmatrix}$ $x_1 = \frac{5}{2}$, $x_3 = \frac{13}{4}$ $Z = 4\left(\frac{5}{2}\right) + 7\left(\frac{13}{4}\right) = 32.75$ //

⋮

Theorem: Definition of Hyperplanes

A hyperplane in R^n is defined as

$$c^T x = z$$

Where x is a vector in R^n and z is a constant

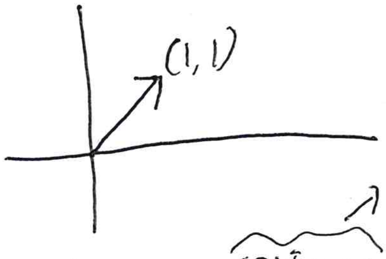
Every single vector on the Hyperplane is orthogonal to the vector C

Hyperplane in 2D

2x+2y=4

Remember that the Dote product of two vectors normal to each other must be zero

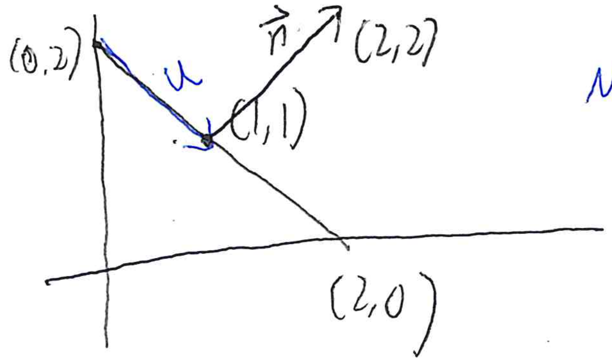
position vector



$$2x+2y=4 \Rightarrow \begin{pmatrix}2 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=4 \Rightarrow \underbrace{\begin{pmatrix}2\\2\end{pmatrix}^T\begin{pmatrix}x\\y\end{pmatrix}}_{\text{dote product of } \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}}=4$$

any position vector in the set $2x+2y=4$ has a dote product of 4 with the vector (2,2)

consider a vector $\vec{u}$ on th set; and (2,2) be $\vec{n}$



$u = \begin{pmatrix}-1\\1\end{pmatrix}$,

$u \cdot \vec{n} = 0$

Generally for 2 points $\begin{pmatrix}x_1\\y_1\end{pmatrix}$ $\begin{pmatrix}x_2\\y_2\end{pmatrix}$ in the set.

$\vec{u} = \begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix}$ $\cancel{\vec{u}\,\vec{n} = \begin{pmatrix}x_1\\y_1\end{pmatrix}\vec{n}}$

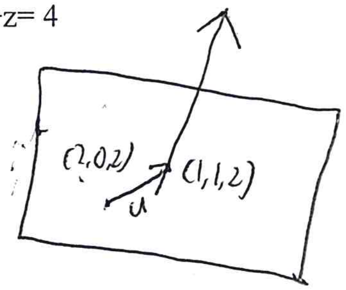
$\vec{u}\cdot\vec{n} = \begin{pmatrix}x_1\\y_1\end{pmatrix}\vec{n} - \begin{pmatrix}x_2\\y_2\end{pmatrix}\vec{n} = 4-4=0$.

any vector $\vec{u}$ in the hyperplane

$\vec{u}\cdot\vec{n} = 0$

Hyperplane in 3D

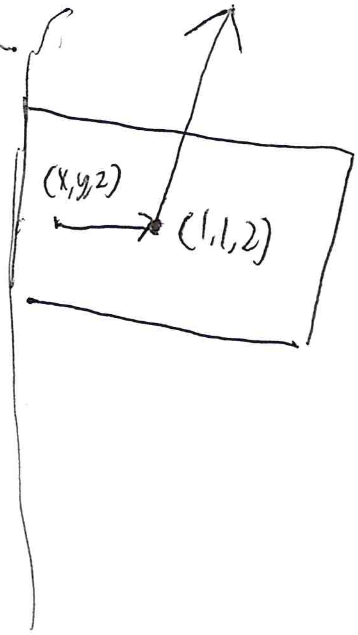
x+y+z= 4



$$u = \begin{pmatrix} 1-2 \\ 1-0 \\ 2-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$u \cdot \vec{n} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$$

apply the other defination.



$$\begin{pmatrix} x-1 \\ y-1 \\ z-2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x+y+z=4.$$

Example 1: Given that the points ~~(1,3,4),(2,1,6),(2,1,8)~~ is on the same hyperplane, find the formula of the hyperplane. (0,1,0), (1,0,-1), (0,0,-3).

$$u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \qquad \vec{n} = \begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ -1 & 0 & -2 \end{vmatrix}$$

$$u \times v = \vec{n} \qquad = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} x-1 \\ y-0 \\ z+1 \end{pmatrix} \qquad \begin{pmatrix} x-1 \\ y-0 \\ z+1 \end{pmatrix} \cdot \vec{n} = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$2x+3y-z=3$$

Operations on set:

Basic proving techniques in sets:

To prove A to be a subset of B, we generally prove towards the direction that for any element x belongs to A, it belongs to B as well

Example 2:

$A \subseteq B \quad C \subseteq D$

$x \in A,\ x \in B$

$x \in C,\ x \in D$

show that $A \cap C \subseteq B \cap D$

$x \in A \cap C \iff x \in A$ and $x \in C$

$\iff x \in B$ and $x \in D$

$\iff x \in B \cap D$

To prove two set being equal, a well known technique is to prove that they are subset of each other.

$A = B \iff A \subseteq B$ and $B \subseteq A$

try Excercise 1.

Closure of a set:

Some Definition:

$\forall$ = for all , $\exists$ = there exist.

~~∂S~~ = · Boundary point of a set A. $x_1 \exists$ $\forall$ open ball B with radius r, center at x_1, $\exists\ x_i \in A, \in B$; $\exists\ x_j \notin A, \in B$

∂S = collection of all boundary point

$\bar{S} = \partial S \cap S$, $S^\circ = S / \partial S$

~~Close set~~ S is close if $\partial S \subseteq S$, Open if $\partial(\mathbb{R}^n - S) \subseteq \mathbb{R}^n - S$

A very simple example:

$(1,3)$ any open interval

$\partial S = \{1\}, \{3\}$ $\quad \bar{S} = [1,3]$

Some facts:

1) S is close $\Leftrightarrow S = \bar{S}$
2) S is open $\Leftrightarrow S = S^\circ$
3) $\bar{S}$ is close
4) S° is open

Proves of 3)

target: $\partial\bar{S} \subset \bar{S}$

$\bar{S}$ can be written as $S \cup \partial\bar{S}$ for some S

$S \cup \partial S = S^\circ \cup \partial S \Rightarrow \partial S = \partial\bar{S}$ $\quad \partial\bar{S} \subset \bar{S}$

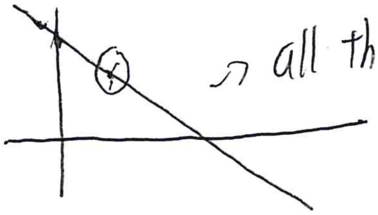
Proves of 4) Target: $\partial S \subset S^\circ / \mathbb{R}^n$

$S^\circ = S / \partial S$, i.e $\partial S \notin S^\circ$

$\partial S \in S^\circ / \mathbb{R}^n = S^\circ - \mathbb{R}^n$

A 2D Hyper plane must be closed

Thinking



↗ all the points on the plane is a boundary point.

Convex set:

A set C is said to be convex if all x_1 and x_2 in C and $0 < \lambda < 1$, $x_1\lambda + x_2(1-\lambda)$ belongs to C as well

Fact 1:

Any close/open interval is convex:

$\lambda x_1 + (1-\lambda)x_2$

$[\ x_1 \quad\quad x_2\ ]$

$a \quad\quad\quad\quad b$

Fact 2:

Any finite intersection of close set is convex

consider $S_1, \cdots, S_N$ are all convex

i.e $\forall n$, if $x_1 \in S_n$, $[\lambda x_1 + (1-\lambda)x_1] \in S_n$.

now if $x \in \bigcap_N S_n$, $x \in$ all $S_1 \cdots S_N$

$\lambda x_1 + (1-\lambda)x_1 \in$ all $S_1 \cdots S_N . \in \bigcap_N S_n$ //.

Example 3:

Union of convex set may not be convex

consider $[1,3] \cup [4,6]$

$[\quad]\quad\quad [\quad]$

$1 \quad 3 \quad\quad 4 \quad 6$

Example 4:

finit.

If there's an ascending sequence of convex set, then the union of the sequence must be convex if $A_1, \cdots, A_n$ ar all convex.

$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n$

$\bigcup_{N} A_n = A_N \to$ convex.

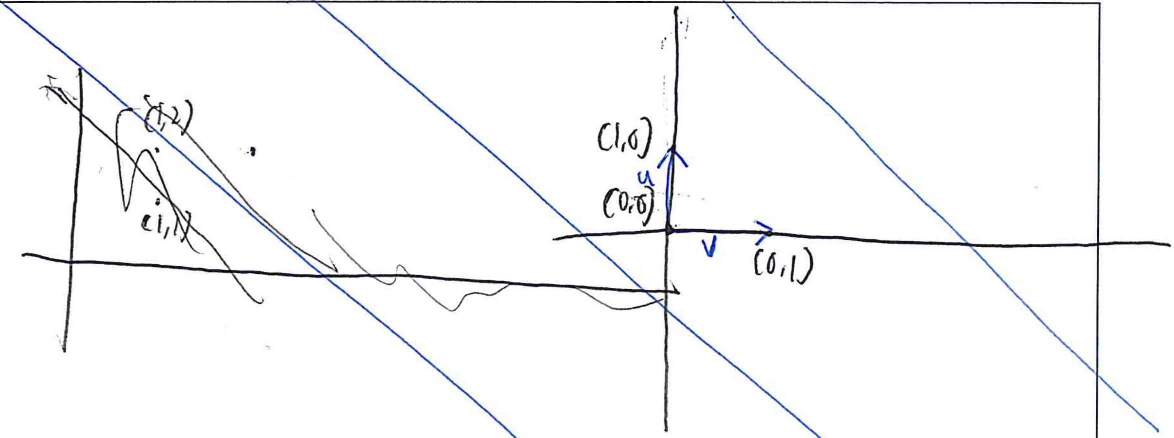
(Try to prove for the case of intersection)

Convex combination:

a convex combination of $x_1, \cdots, x_n$ is $\varphi_1 x_1 + \varphi_2 x_2 + \cdots + \varphi_n x_n$ s.t. $\varphi_1 + \cdots + \varphi_n = 1$ and $0 < \varphi_1, \varphi_2, \cdots, \varphi_n < 1$

Convex combination of vectors:

Skip.



Example5: If a set of points are on a particular hyper plane, then any convex combination of the set of points are on the hyper plane:

if ~~$a_1 x_1 + a_2 x_2$~~

$X_1, \dots, X_N$ satisfied the condition that

$X_n \cdot \vec{n} = C$ ~~$\forall n \in N$~~

for $n = 1, \dots, N$.

$(\varphi_1 X_1 + \dots + \varphi_N X_N) \cdot \vec{n}$

$= \varphi_1 (X_1 \cdot \vec{n}) + \dots + \varphi_N (X_N \cdot \vec{n}) = a_1 C + \dots + \varphi_n C$

$= C$ //.

~~th~~e

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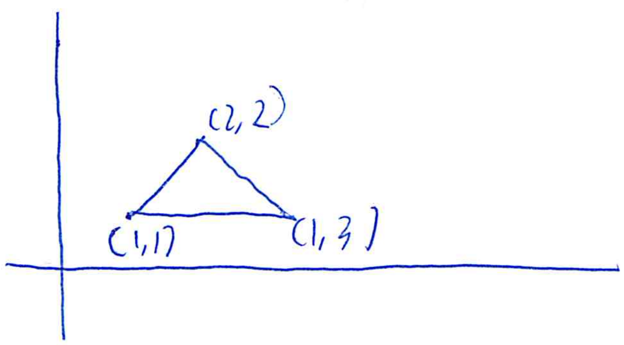
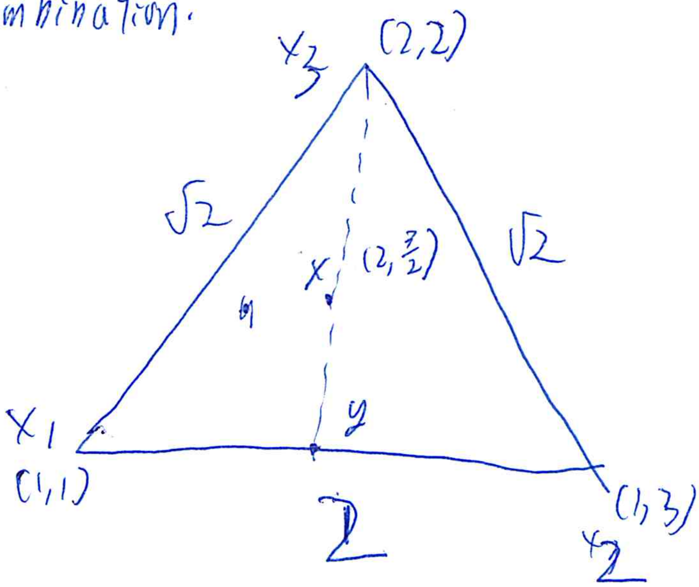
before example 5: Convex combination.





$y = pX_1 + qX_2$

$p = \frac{|X_2 y|}{|X_2 X_1|}$ $q = \frac{|X_1 y|}{|X_2 X_1|}$, $p = q = \frac{1}{2}$

$l = \frac{|xy|}{|X_3 y|}$ $k = \frac{|X X_3|}{|X_3 y|}$, $l = k = \frac{1}{2}$

$X = kpX_1 + kqX_2 + lX_3$

$= \frac{1}{4} X_1 + \frac{1}{4} X_2 + \frac{1}{2} X_3$ //

Example6: Consider a LPP, if a set of points in the feasible sets give you the same numeric value, then any convex combination of the set of points will give you the same value.

consider the ~~feasible~~ set of points to be ~~$x_1, \cdots, x_N$~~ $s_1, s_2, \cdots, s_N$

and the LPP to be

max/min $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = C$

Here Note that

$$s_1 = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_n^1 \end{pmatrix} \quad \cdots \quad s_N = \begin{pmatrix} x_1^N \\ x_2^N \\ \vdots \\ x_n^N \end{pmatrix}$$

Now if all give the same numerical value to the LPP, then.

$$\left.\begin{aligned} a_1 x_1^1 + a_2 x_2^1 + a_3 x_3^1 + \cdots + a_n x_n^1 &= K \\ &\vdots \\ a_1 x_1^N + a_2 x_2^N + \cdots + a_n x_n^N &= K \end{aligned}\right\} \text{on the same hyper plane.}$$

from last question, any convex combination will be on the hyper plane, i.e must give the same value.